Linear Regression Applied to System Identification for Adaptive Control Systems

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The purpose of this paper is to describe a method of process identification using a linear regression technique and to indicate how this method may be applied to adaptive control systems. The unknown system parameters are considered as additional state variables. Estimates of system parameters as well as the system state are made from noise-contaminated data. Differential equations with random forcing functions describing the parameter variations are adjoined to the system of differential equations describing the process. The estimation of the error is assumed to propagate linearly about the current estimates, which are updated as new data are received.

Introduction

THE adaptive control problem has received considerable attention over the past several years. Basically, this is the problem of controlling a process (in this paper the process will be an aerospace vehicle) where imperfect or limited information is available describing the vehicle parameters that change considerably during the interval in which control is required.

This paper describes a statistical method for estimating space vehicle parameters as well as vehicle orientation from noise-contaminated data. Furthermore, it will be shown how this method might be applied to an adaptive control system. A second-order linear system, representative of an attitude rate control system for a space vehicle, will be used to illustrate the method. However, this method is not theoretically limited to second-order systems or even linear systems. The use of a linear regression technique applied to process identification was motivated by some of the recent results obtained by Kalman[1] in linear filtering. This technique appears very attractive, since no extraneous system inputs are needed. A further advantage of this method is that it is not limited to steady-state analysis as are many previously suggested schemes. A second-order system with unknown time-varying parameters is used to characterize the vehicle's attitude response. (The unknown parameters in this case would be the system damping, frequency, and gain.) The system parameters are considered as additional state variables, and differential constraints are adjoined to these variables to provide correlation between present estimates and past estimates of system parameters. A priori estimates of the vehicle parameters, that is, estimates of the behavior of the system exclusive of measurement data, are included in the adjoined differential equations describing the vehicle's parameter behavior. Estimates based on measurement data and control inputs are made for both vehicle orientation and vehicle parameters. The statistical degree of freedom is provided by introducing random variables as forcing functions in the adjoined equations for the unknown parameters.

The estimation of the vehicle parameters first will be analyzed as a sampled-data problem, linearizing the system behavior between sampling intervals. A linear regression technique is used to derive a recursive relationship for the updated estimates of vehicle parameters and orientation as a function of the last estimates and new measurement data. This approach avoids the need for large computer storage and leads directly to a filtering concept. As the sampling interval approaches zero in the limit, the recursive relationships reduce to differential equations describing time-varying nonlinear filters for the estimators of the vehicle parameters and orientation.

The basic concepts associated with this method are quite simple; the estimation of the error is assumed to propagate by a linearization of a nonlinear system (the system becomes nonlinear when one adjoins the differential equations describing the unknown parameter variations) about the current estimate of the system variables. Using these linear equations allows the estimates to be updated with new data as in Ref. 1.

The experimental results that are included in this paper are regarded as encouraging since they are quite insensitive to the assumptions made, a characteristic very desirable for engineering applications.

Problem Formulation

To illustrate how a linear regression technique can be applied to system identification for use in adaptive systems, a specific example representative of an attitude rate control system for a space vehicle will be used. It will be assumed, as is seen in Fig. 1, that the vehicle dynamics can be represented by a second-order linear system with parameters δ, ω_n, and k, which are not precisely known and which vary considerably during the interval of time when control is to be exerted. The transfer function notation, although not applicable to time-varying systems, is used in Fig. 1 merely to illustrate the problem being considered.

A realistic approach to the problem is made by assuming that the measured pitch rate $z(t)$, which will be designated as data, is the actual pitch rate $\dot{\theta}(t)$ contaminated with measurement noise $v(t)$. Furthermore, it is assumed that the actual control signal $u(t)$ is the desired control signal $u^*(t)$ plus additive noise $\delta u(t)$. Since the identification problem is of primary concern, that is, estimating the parameters $\delta(t)$, $\omega_n(t)$, and $k(t)$, as well as $\dot{\theta}(t)$ and $\ddot{\theta}(t)$, from the measured data $z(t)$, consideration of the control or "actuation problem" can be deferred until later. It is tacitly assumed that the two problems are separable, as discussed in Refs. 2 and 3.

Although the proposed method is not limited to linear systems, one can proceed to develop the method using the linear example being discussed. It is convenient to define the

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variable $x_1 \equiv d\theta/dt$. The system of differential equations describing the motion of the vehicle about the pitch axis is then written as

$$\begin{aligned} dx_1/dt &= x_2 \\ dx_2/dt &= a_1(t)x_2 + a_2(t)x_1 + a_3(t)u(t) \end{aligned} \tag{1}$$

where the unknown time-varying coefficients $a_1(t)$, $a_2(t)$, and $a_3(t)$ are defined as

$$a_1(t) \equiv -2\delta(t)\omega_n(t)$$
$$a_2(t) \equiv -\omega_n^2(t)$$
$$a_3(t) \equiv k(t)$$

Initial conditions for Eq. (1) are drawn from a set of normally distributed random variables of known statistical properties. It is assumed that estimates of $a_i(t)$, $i = 1 \ldots 3$, are equally as good as estimates of $\delta(t)$, $\omega_n(t)$, and $k(t)$ for the determination of the desired control signal $u^*(t)$.

Measured data $z(t)$ are the sum of the pitch rate $x_1(t)$ and additive noise:

$$z(t) = x_1(t) + v(t) \tag{2}$$

The noise $v(t)$ is assumed to be a normally distributed random variable with known statistical properties:

$$\begin{aligned} E[v(t)] &= 0 \\ E[v(t)v(\tau)] &= \sigma_v^2(t)\delta(t-\tau) \end{aligned} \tag{3}$$

where $\delta(t-\tau)$ is the Dirac delta, and $\sigma_v^2(t)$ is given. The case where the control signal $u(t)$ is the sum of a desired control signal $u^*(t)$ and additive noise is considered:

$$u(t) = u^*(t) + \delta u(t) \tag{4}$$

The additive noise $\delta u(t)$ is assumed to be a normally distributed random variable modulated by a function of the desired control signal:

$$\delta u(t) = S[u^*(t)]w_0(t) \tag{5}$$

where the statistical properties of $w_0(t)$ are given:

$$\begin{aligned} E[w_0(t)] &= 0 \\ E[w_0(t)w_0(\tau)] &= \sigma_{w0}^2\delta(t-\tau) \end{aligned} \tag{6}$$

Certain assumptions and approximations now will be made which we feel are reasonable, but which we will not attempt to justify. Some a priori knowledge of the expected values of $a_i(t)$ may be available which we would like to include in the analysis. These estimates, if they are available, are defined as $\bar{a}_i(t)$. It also is assumed that the coefficients $a_i(t)$ vary continuously in a random manner. With this in mind, the set of differential equations is adjoined:

$$da_i/dt = \alpha_i(t)[a_i(t) - \bar{a}_i(t)] + w_i(t) \qquad i = 1 \ldots 3 \tag{7}$$

to Eqs. (1). The differential constraints insure continuity, whereas the normally distributed random functions of time $w_i(t)$ provide the needed statistical degree of freedom. Initial conditions for Eqs. (7) are drawn from a set of normally distributed random variables. The parameters $\alpha_i(t)$ are assumed given, as are the statistical properties of $w_i(t)$:

$$\begin{aligned} E[w_i(t)] &= 0 \\ E[w_i(t)w_i(\tau)] &= \sigma_{wi}^2\delta(t-\tau) \qquad i = 1 \ldots 3 \end{aligned} \tag{8}$$

One might refer to the set of Eqs. (7) as a statistical model of the vehicle parameters, but it is not the only model that could be chosen.

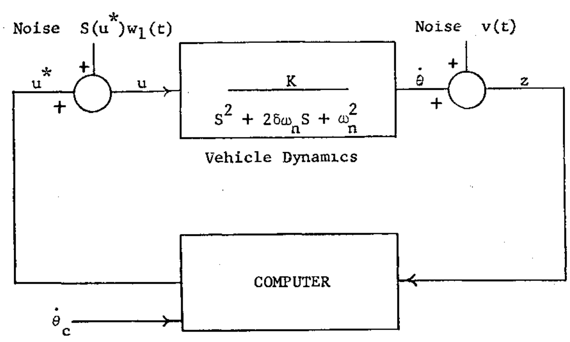

Fig. 1 Attitude rate control system.

Sampled Data Analysis

For the sampled data analysis, time will be assumed to be divided into discrete intervals of length Δt. It further will be assumed that the control is constant over any one interval of time changing in a stepwise manner between intervals. Data are sampled at the end of each interval. For convenience, one can designate a variable at time $n\Delta t$ by $x(n)$. One also will need notation to designate the estimate of a variable at time $n\Delta t$, given all past data and desired control inputs to time $k\Delta t, n \geq k$. This will be denoted as

$\hat{x}(n|k) \equiv$ estimate of $x(n)$ given $[z(1), \ldots, z(k), u^*(1), \ldots, u^*(k)]$

These will be the fundamental quantities, with $k = n - 1$ or n; however, one also will need estimates between sampling instants which can be designated as $\hat{x}(t|n)$:

$\hat{x}(t|n) \equiv$ estimate of $x(t)$ given $[z(1), \ldots, z(n), u^*(1), \ldots, u^*(n)]$ $\quad t \geq n\Delta t$

and similarly for $a_i(t)$. Then one can define an error, e.g.,

$$\delta x(t|n) = x(t) - \hat{x}(t|n) \qquad t \geq n\Delta t \tag{9}$$

and again similarly for $\delta a_i(t)$.

One can estimate between sampling intervals by means of

$$\begin{aligned} \frac{d\hat{x}_1(t|n)}{dt} &= \hat{x}_2(t|n) \\ \frac{d\hat{x}_2(t|n)}{dt} &= \hat{a}_1(t|n)\hat{x}_2(t|n) + \hat{a}_2(t|n)\hat{x}_1(t|n) + \hat{a}_3(t|n)u^*(t) \\ \frac{d\hat{a}_2(t|n)}{dt} &= \alpha_i(t)[\hat{a}_i(t|n) - \bar{a}_i(t|n)] \end{aligned} \tag{10}$$

Substituting Eq. (9), etc., into Eqs. (1) and (7) and subtracting Eqs. (10), neglecting terms of second order, one has, using a vector matrix notation,

$$\frac{d\boldsymbol{\delta}\mathbf{y}(t|n)}{dt} = \mathfrak{L}(t|n)\boldsymbol{\delta}\mathbf{y}(t|n) + \boldsymbol{\Omega}(t) \tag{11}$$

where the vectors

$$\boldsymbol{\delta}\mathbf{y}^T(t|n) \equiv [\delta x_1(t|n)\ \delta x_2(t|n)\ \delta a_1(t|n)\ \delta a_2(t|n)\ \delta a_3(t|n)]$$

and

$$\boldsymbol{\Omega}^T(t) \equiv [0\ \hat{a}_3(t|n)S(u^*)w_0(t)\ w_1(t)\ w_2(t)\ w_3(t)]$$

and the matrix

$$\mathfrak{L}(t|n) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \hat{a}_2(t|n) & \hat{a}_1(t|n) & \hat{x}_2(t|n) & \hat{x}_1(t|n) & u^*(t) \\ 0 & 0 & \alpha_1(t) & 0 & 0 \\ 0 & 0 & 0 & \alpha_2(t) & 0 \\ 0 & 0 & 0 & 0 & \alpha_3(t) \end{bmatrix}$$

The superscript T denotes the transpose. The $\mathfrak{L}(t|n)$ matrix is seen to be composed of the estimates of parameters $a_i(t)$, as

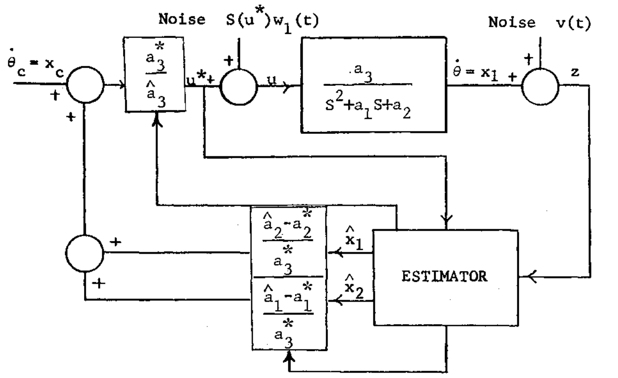

Fig. 2 Adaptive system using a linear regression method for the estimator.

well as estimates of the vehicle state $x_1(t)$ and $x_2(t)$ at time t, given data to time $n\Delta t$. Equation (11) describes the propagation of the errors between sampling intervals. At time $n\Delta t$, one can assume that the error $\boldsymbol{\delta}\mathbf{y}(n|n)$ is distributed multinormally with given characteristics

$$E[\boldsymbol{\delta}\mathbf{y}(n|n)] = 0$$
$$E[\boldsymbol{\delta}\mathbf{y}(n|n)\boldsymbol{\delta}\mathbf{y}^T(n|n)] = P(n|n) \tag{12}$$

This is certainly true initially by assumption.

For convenience, let a vector for the estimates of the state and parameters be defined as

$$\mathbf{y}(n|n) \equiv [x_1(n|n)\ x_2(n|n)\ a_1(n|n)\ a_2(n|n)\ a_3(n|n)]$$

One now has a concise formulation of the problem. We wish to estimate $\boldsymbol{\delta}\mathbf{y}(n+1)$ conditioned on data at $(n+1)\Delta t$, given that $\hat{\boldsymbol{\delta}}\hat{\mathbf{y}}(n|n)$ is zero with known variance. The estimate of $\mathbf{y}(n+1)$ is given by

$$\hat{\mathbf{y}}(n+1|n+1) = \hat{\mathbf{y}}(n+1|n) + \hat{\boldsymbol{\delta}}\hat{\mathbf{y}}(n+1|n+1) \tag{13}$$

We will now proceed to discuss how one calculates $\hat{\boldsymbol{\delta}}\mathbf{y}(n+1|n)$, $\hat{\boldsymbol{\delta}}\mathbf{y}(n+1|n+1)$ and the variance of $\boldsymbol{\delta}\mathbf{y}(n+1|n+1)$, which determines the initial conditions for the next interval of time.

One can assume that the vector $\boldsymbol{\Omega}(t)$ is constant in the interval $n\Delta t \leq \tau < (n+1)\Delta t$ with components selected from a multinormal distribution of known statistical properties:

$$E[\boldsymbol{\Omega}_n] = 0 \qquad E[\boldsymbol{\Omega}_n\boldsymbol{\Omega}_k^T] = Q_n\delta_{nk} \tag{14}$$

where δ_{nk} is the Kronecker delta. The definitions of a sampled random function of the type defined in Eq. (8) must be examined (see Feller,[4] p. 324). Specifically, in order to maintain consistent behavior for the output of a linear system (see Kalman[1]), where, for the continuous function,

$$E[\boldsymbol{\Omega}(t)\boldsymbol{\Omega}^T(t)] = Q(t)\delta(t-\tau) \tag{15}$$

then,

$$\lim_{\Delta t \to 0} Q_n\Delta t = Q(t) \tag{16}$$

Similarly for $v(t)$. For the specific example,

$$Q(t) \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a_3^2S^2(u)\sigma_{w0}^2(t) & 0 & 0 & 0 \\ 0 & 0 & \sigma_{w1}^2(t) & 0 & 0 \\ 0 & 0 & 0 & \sigma_{w2}^2(t) & 0 \\ 0 & 0 & 0 & 0 & \sigma_{w3}^2(t) \end{bmatrix}$$

By use of the transition matrix,[5] the solution of Eq. (11) is written:

$$\boldsymbol{\delta}\mathbf{y}(n+1) = \Phi(n+1,n)\boldsymbol{\delta}\mathbf{y}(n) + \Gamma(n+1,n)\boldsymbol{\Omega}(n) \tag{17}$$

To obtain $\Phi(n+1,n)$, $\Gamma(n+1,n)$ in Eq. (17), one first must solve Eqs. (10), from which one also obtains $\mathbf{y}(n+1|n)$. An estimate of $\boldsymbol{\delta}\mathbf{y}(n+1)$, given data to $(n+1)\Delta t$, is made by linear regression, i.e.,

$$\hat{\boldsymbol{\delta}}\hat{\mathbf{y}}(n+1|n+1) = \Psi(n+1)\tilde{z}(n+1) \tag{18}$$

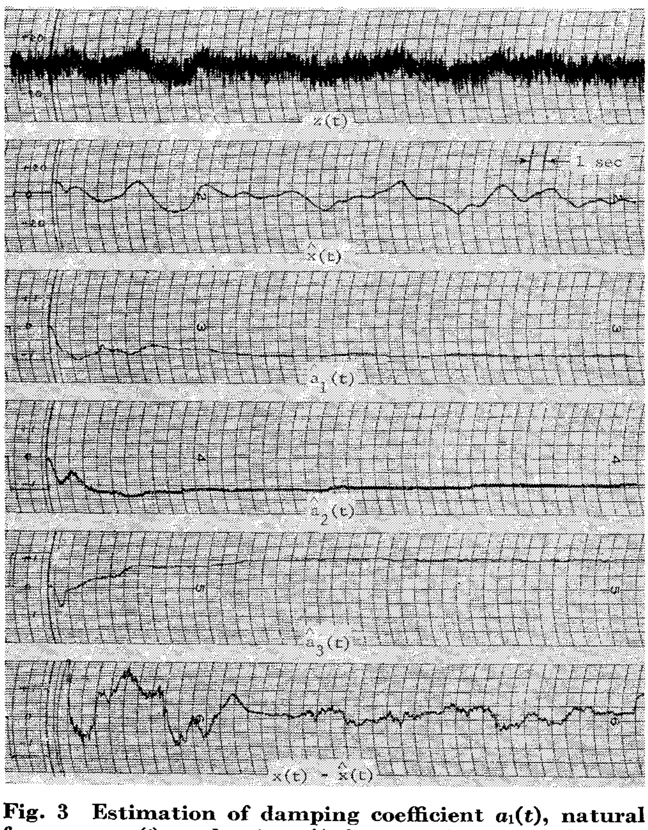

Fig. 3 Estimation of damping coefficient $a_1(t)$, natural frequency $a_2(t)$, and gain $a_3(t)$ for a random input [$a_1(t) = -1$, $a_2(t) = -1$, $a_3(t) = 1$, $\hat{a}_1(0) = 0$, $\hat{a}_2(0) = 0$, $\hat{a}_3(0) = 0$].

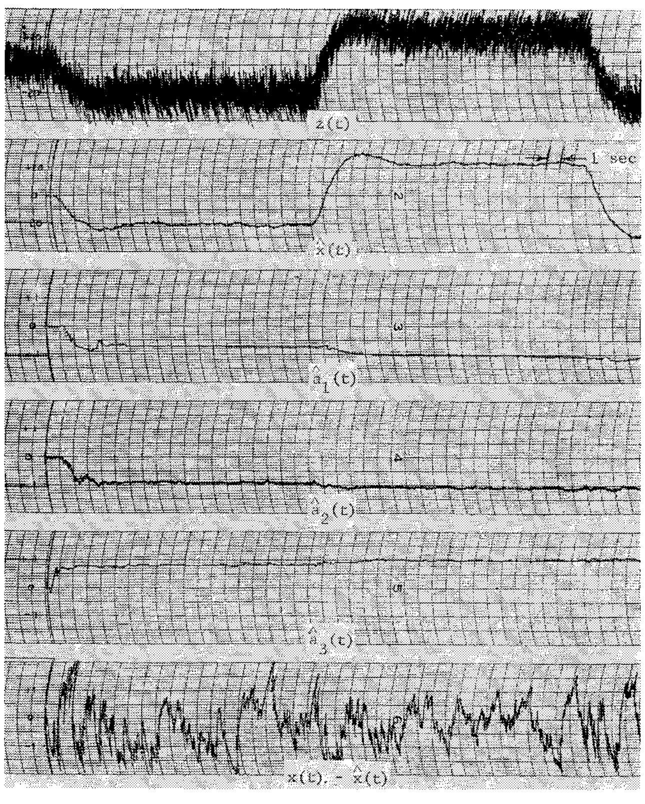

Fig. 4 Estimation of damping coefficient $a_1(t)$, natural frequency $a_2(t)$, and gain $a_3(t)$ for a step input [$a_1(t) = -1$, $a_2(t) = -1$, $a_3(t) = 1$, $\hat{a}_1(0) = 0$, $\hat{a}_2(0) = 0$, $a_3(0) = 0$].

where $\tilde{z}(n + 1)$ are new data, defined as the difference between the actual data received and the estimate of the data conditioned on the previous sampling instant:

$$\begin{aligned}\tilde{z}(n+1) &= z(n+1) - \hat{z}(n+1|n) = \\ &\quad x_1(n+1) + v(n+1) - \hat{x}_1(n+1|n) \\ &= \delta x(n+1|n) + v(n+1)\end{aligned} \tag{19}$$

To write Eq. (19) in terms of the y vector and preserve dimensionality, the row vector is introduced:

$$\mathbf{M} \equiv [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

then

$$\tilde{z}(n+1) = z(n+1) - \mathbf{M}\hat{\mathbf{y}}(n+1|n) \tag{20a}$$

$$= \mathbf{M}\delta\mathbf{y}(n+1|n) + v(n+1) \tag{20b}$$

The column vector $\Psi(n + 1)$ is determined by minimizing

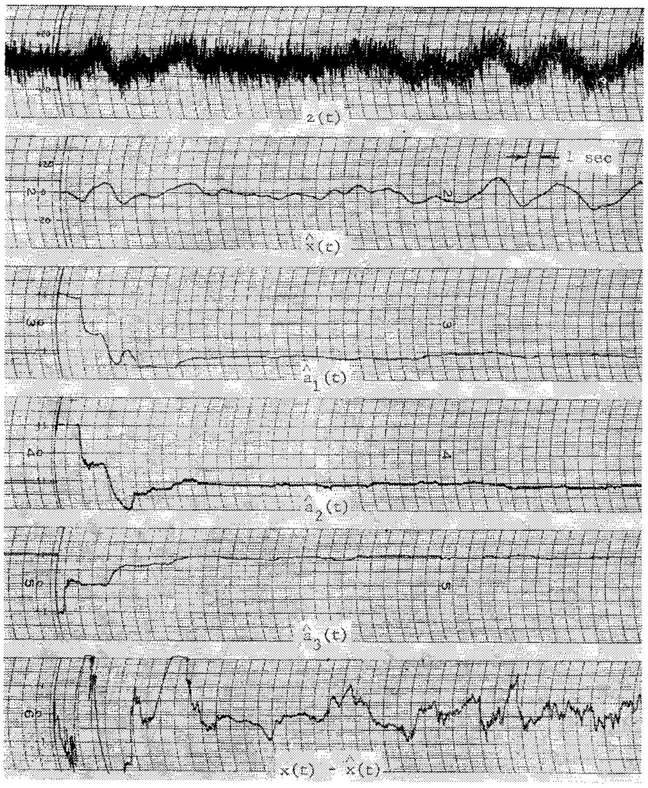

Fig. 5 Estimation of damping coefficient $a_1(t)$, natural frequency $a_2(t)$, and gain $a_3(t)$ with initial estimates of opposite sign [$a_1(t) = -1$, $a_2(t) = 1$, $a_3(t) = 1$, $\hat{a}_1(0) = 1$, $\hat{a}_2(0) = 1$, $\hat{a}_3(0) = -1$].

the diagonal elements of the covariance matrix $P(n + 1|n + 1)$ of the error $\delta\mathbf{y}(n + 1|n + 1)$:

$$\Psi(n+1) = P(n+1|n)\mathbf{M}^T[\mathbf{M}P(n+1|n)\mathbf{M}^T + \sigma_v^2(n+1)]^{-1} \tag{21}$$

where

$$P(n+1|n) = \Phi(n+1,n)P(n|n)\Phi(n+1,n) + \Gamma(n+1,n)Q(n)\Gamma^T(n+1,n) \tag{22}$$

and

$$P(n+1|n+1) = [I - \Psi(n+1)\mathbf{M}]P(n+1|n) \tag{23}$$

At $n = 0$, which is the start of the process, initial values of $\hat{\mathbf{y}}(0)$ and $P(0|0)$ are given. That is, for the specific example, initial estimates of $x_1(0)$, $x_2(0)$, and $a_i(0)$ are given, along with the variances of their respective errors. From Eqs. (21–23), $P(1|0)$, $P(1|1)$, and $\Psi(1)$ are calculated. When data are received at the first sampling interval, $\delta\hat{x}_1(1|1)$,

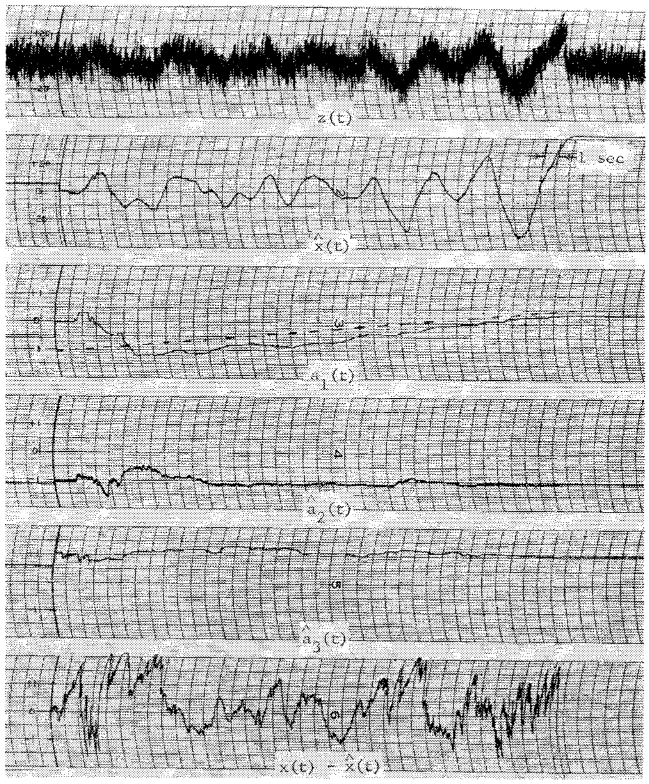

Fig. 6 Estimation of a time-varying damping coefficient $a_1(t)$ [$a_1(t)$ is time varying, $a_2(t) = -1$, $a_3(t) = 1$].

$\delta\hat{x}_2(1|1)$, and $\delta a_i(1|1)$ are calculated from Eq. (18), and the estimates of the respective variables are determined from Eq. (13). These values form initial conditions for the next interval, and the calculation proceeds. The important relationships are summarized as

$$\begin{aligned}P(n+1|n) &= \Phi(n+1,n)P(n|n)\Phi^T(n+1,n) + \\ &\qquad \Gamma(n+1,n)Q(n)\Gamma^T(n+1,n) \\ \Psi(n+1) &= P(n+1|n)\mathbf{M}^T[\mathbf{M}P(n+1|n)\mathbf{M}^T + \\ &\qquad \sigma_v^2(n+1)]^{-1} \\ \hat{\mathbf{y}}(n+1|n) &\qquad \text{[obtained from Eqs. (10)]} \\ \delta\hat{y}(n+1|n+1) &= \Psi(n+1)[z(n+1) - \\ &\qquad \mathbf{M}\hat{\mathbf{y}}(n+1|n)] \\ \hat{\mathbf{y}}(n+1|n+1) &= \hat{\mathbf{y}}(n+1 \mid n) + \\ &\qquad \delta\hat{\mathbf{y}}(n+1 \mid n+1) \\ P(n+1|n+1) &= [I - \Psi(n+1)\mathbf{M}]P(n+1|n)\end{aligned} \tag{24}$$

Continuous Analysis

To analyze the continuous case, that is, where one receives data continuously, one allows the sampling interval Δt to approach zero in the limit. A state vector $x(t)$ is defined as

$$\mathbf{x}^T(t) \equiv [x_1(t)x_2(t)]$$

and the estimate of $\mathbf{x}$ at $(n + 1)\Delta t$, given data to $n\Delta t$ in terms of the transition matrix for the vehicle $\Phi^x(n + 1,n)$, is written as

$$\hat{\mathbf{x}}(n+1|n) = \Phi^x(n+1,n)\hat{\mathbf{x}}(n|n) + \Gamma^x(n+1,n)u^*(n) \tag{25}$$

The estimate of x at $(n + 1)\Delta t$, given data to $(n + 1)\Delta t$, is therefore

$$\hat{\mathbf{x}}(n+1|n+1) = \Phi^x(n+1,n)\hat{\mathbf{x}}(n|n) + \Gamma^x(n+1,n)u^*(n|n) + \delta\hat{\mathbf{x}}(n+1|n+1) \tag{26}$$

Subtracting $\hat{\mathbf{x}}(n|n)$ from each side of Eq. (26), dividing by

Δt, and passing to the limit gives

$$\frac{\mathbf{d}\hat{\mathbf{x}}(t)}{dt} = \hat{A}(t)\hat{\mathbf{x}}(t) + \hat{B}(t)u^*(t) + \lim_{\Delta t\to 0} \boldsymbol{\delta}\hat{\mathbf{x}} \frac{(t+\Delta t|t+\Delta t)}{\Delta t} \qquad (27)$$

where

$$\hat{A}(t) \equiv \begin{bmatrix} 1 & 0 \\ \hat{a}_1(t) & \hat{a}_2(t) \end{bmatrix} \qquad \hat{B}(t) \equiv \begin{bmatrix} 0 \\ \hat{a}_3(t) \end{bmatrix}$$

Here it is understood that $\hat{\mathbf{x}}(t)$ designates the estimates of $\mathbf{x}(t)$, given data to t, and likewise for $\hat{A}(t)$ and $\hat{B}(t)$. Similarly, a vector $\boldsymbol{\xi}(t)$ is defined as

$$\boldsymbol{\xi}(t) \equiv [a_1(t)\ a_2(t)\ a_3(t)]$$

and the following is obtained:

$$\frac{\mathbf{d}\hat{\boldsymbol{\xi}}(t)}{dt} = C(t)[\hat{\boldsymbol{\xi}}(t) - \bar{\boldsymbol{\xi}}(t)] + \lim_{\Delta t\to 0} \boldsymbol{\delta}\hat{\boldsymbol{\xi}} \frac{(t+\Delta t|t+\Delta t)}{\Delta t} \qquad (28)$$

where

$$C(t) \equiv \begin{bmatrix} \alpha_1(t) & 0 & 0 \\ 0 & \alpha_2(t) & 0 \\ 0 & 0 & \alpha_3(t) \end{bmatrix}$$

Carrying through this limiting process, one sees that the $P(n+1|n)$ and $P(n+1|n+1)$ matrices become the same and will be designated by $P(t)$. Equations (27) and (28) for the updated estimates reduce to

$$\mathbf{d}\hat{\mathbf{x}}(t)/dt = \hat{A}(t)\hat{\mathbf{x}}(t) + \hat{B}(t)u^*(t) + P_1(t)M_1^T\sigma_v^{-2}(t)\tilde{z}(t)$$
$$\mathbf{d}\hat{\boldsymbol{\xi}}(t)/dt = C(t)[\hat{\boldsymbol{\xi}}(t) - \bar{\boldsymbol{\xi}}(t)] + P_2(t)M_2^T\sigma_v^{-2}(t)\tilde{z}(t) \qquad (29)$$

where $P_1(t)$ and $P_2(t)$ are partitions of $P(t)$:

$$P_1(t) \equiv \text{cov}\delta\mathbf{x}(t)\ \boldsymbol{\delta}\mathbf{x}^T(t)$$
$$P_2(t) \equiv \text{cov}\delta\boldsymbol{\xi}(t)\ \boldsymbol{\delta}\mathbf{x}^T(t)$$

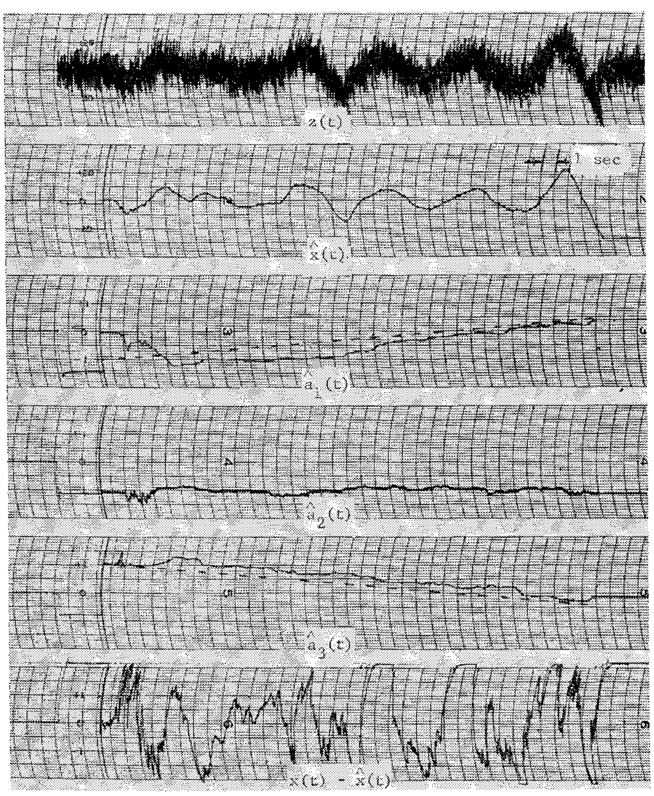

Fig. 7 Estimation of a time-varying damping coefficient $a_1(t)$ and a time-varying gain $a_3(t)$ [$a_1(t)$ is time varying, $a_2(t) = -1$, $a_3(t)$ is time varying].

$$P(t) \equiv \begin{bmatrix} P_1(t) \\ \hline P_2(t) \end{bmatrix}$$

Again $\mathbf{M}_1$ and $\mathbf{M}_2$ are introduced to preserve dimensionality:

$$\mathbf{M}_1 \equiv [1\ 0] \qquad \mathbf{M}_2 \equiv [1\ 0\ 0]$$

The covariance matrix $P(t)$ obeys the matrix equation

$$\dot{P}(t) = \mathcal{L}(t)P(t) + P(t)\mathcal{L}^T(t) - P(t)\mathbf{M}^T(t)\sigma_v^{-2}(t)\mathbf{M}(t)P(t) + Q(t) \qquad (30)$$

Determination of Control Signal

A discussion of how the desired control signal $u^*(t)$ is to be generated has been avoided deliberately until now, because the primary purpose of this paper is to develop a method for measuring (estimating) the vehicle system parameters. However, if this technique is to be applied to adaptive control systems, some discussion is in order. As was pointed out earlier, the separation of the estimation of system parameters and control signal generation is justifiably open to criticism. It would be desirable to formulate the problem such that $u^*(t)$ is determined directly by minimizing the conditional expectation of a system performance measure, such as an integral square criterion. However, upon analysis one soon becomes discouraged with this approach. Therefore, a compromise between what one desires and what one can obtain seems in order. The separation of system parameter estimation and control signal generation certainly has intuitive appeal and motivation. It has been shown[2] for linear systems with known parameters that, for an integral square criterion, the filtering of data can be analyzed separately from the control signal generation, the control law being the same as for the deterministic case with the state variables being replaced with their conditional expectations. With these thoughts in mind, one can proceed to develop a very simple (actually an open-loop) control law to obtain a desired dynamic behavior of the process. Additional feedback would no doubt be required in many applications that would depend on the specific system requirements.

Let it be assumed that $u^*(t)$ is a linear combination of the command attitude rate $\dot{\theta}_c \equiv x_c$, $x_1(t)$, and $\dot{x}_1(t)$, i.e.,

$$u^*(t) = \eta_1 x_c(t) + \eta_2 x_1(t) + \eta_3 \dot{x}_1(t) \qquad (31)$$

It is specified further that the overall system is to respond like a specified model that is described by the differential equations

$$\ddot{x} = a_1^*\dot{x} + a_2^*x + a_3^*x_c \qquad (32)$$

Under these conditions and assumptions,

$$\eta_1 = \frac{a_3^*}{a_3} \qquad \eta_2 = \frac{a_2 - a_2^*}{a_3} \qquad \eta_3 = \frac{a_1 - a_1^*}{a_3} \qquad (33)$$

A typical block diagram of the complete system is shown in Fig. 2.

Simulation

The system discussed was simulated using a combined analog-digital hybrid computing system. This provided what was felt to be a realistic simulation of the problem. The process to be controlled as well as the desired response were simulated on an analog computer, and Eqs. (29) and (30) for the state and parameter estimaters were solved simultaneously on a digital computer using the most elementary integration techniques. The sampling interval was of the order of seven msec. Although the data were sampled, a continuous analysis was used because of the small sampling interval compared to the dynamics of the system.

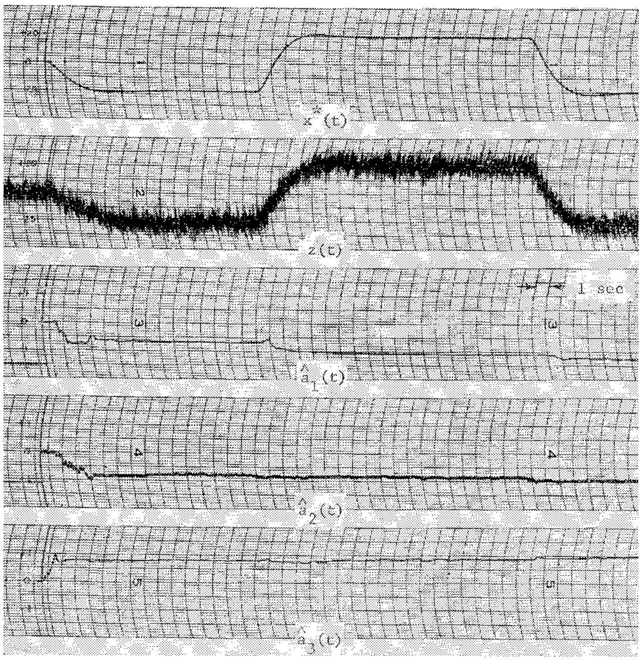

Fig. 8 Control system with desired response 0.7 critical damping, initial estimates of all parameters zero [$a_1^*(t) = -1.4$, $a_2^*(t) = -1$, $a_3^*(t) = 1$, $a_1(t) = -1.4$, $a_2(t) = -1$, $a_3(t) = 1$, $\hat{a}_1(0) = 0$, $\hat{a}_2(0) = 0$, $\hat{a}_3(0) = 0$].

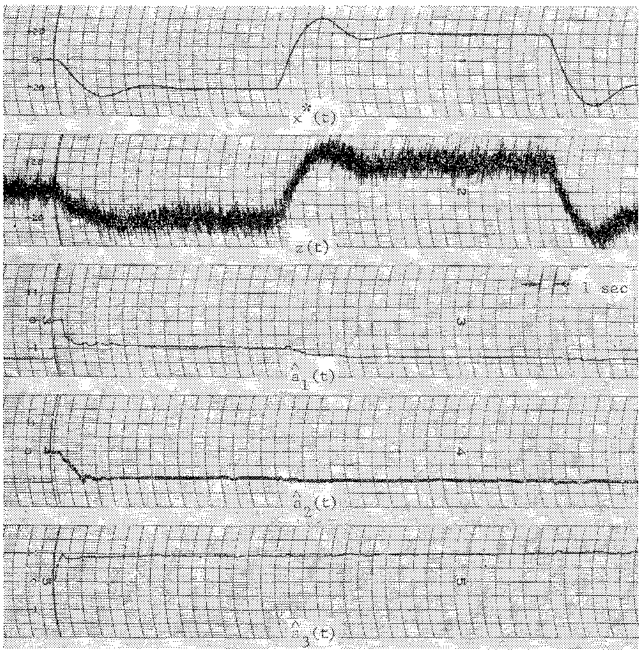

Fig. 9 Control system with desired response 0.35 critical damping, initial estimates of all parameters zero [$a_1^*(t) = -0.7$, $a_2^*(t) = -1$, $a_3^*(t) = 1$, $a_1(t) = -1.4$, $a_2(t) = -1$, $a_3(t) = 1$, $\hat{a}_1(0) = 0$, $\hat{a}_2(0) = 0$, $\hat{a}_3(0) = 0$].

The control signal $u^*(t)$ also was calculated on the digital computer using Eqs. (31) and (33).

Before simulating the entire control system, many open-loop time histories were run off to illustrate process identification for varied conditions. For all conditions, the variance σ_v^2 of the additive noise to the data was assumed to be 25. (The equations have been nondimensionalized.) Perfect information was assumed to be available for the control input $u^*(t)$ and, therefore, $\sigma_{w_c}^2 = 0$.

The values of $\bar{a}_1(t)$, $\bar{a}_2(t)$, and $\bar{a}_3(t)$ were assumed zero, as were α_1, α_2, and α_3 [no a priori information available as to how the parameters $a_1(t)$, $a_2(t)$, and $a_3(t)$ might vary].

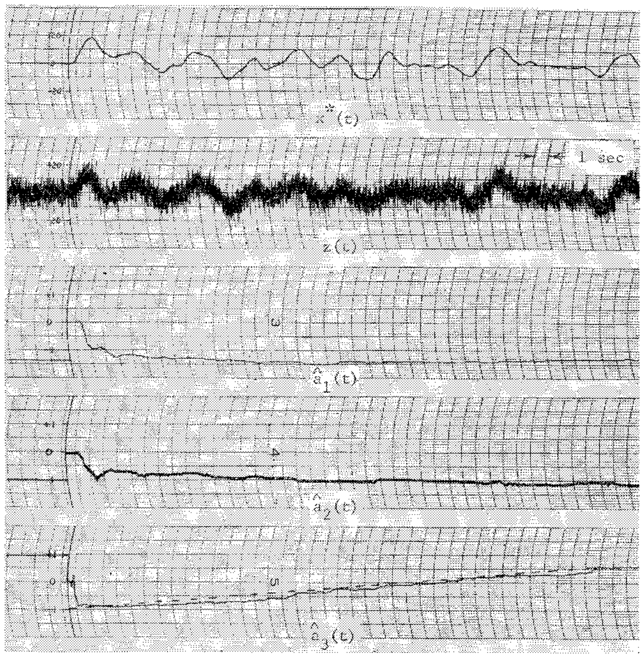

Fig. 10 Control system with desired response 0.7 critical damping, $a_3(t)$ time varying, random input [$a_1^*(t) = -1.4$, $a_2^*(t) = -1$, $a_3^* = 1$, $a_1(t) = -1$, $a_2(t) = -1$, $a_3(t)$ is time varying].

The variances $\sigma_{w_1}^2$, $\sigma_{w_2}^2$, and $\sigma_{w_3}^2$ were all given values of 0.1. Initial estimates of the diagonal elements of the covariance matrix P were all equal to 25 and the off-diagonal initial estimates 0. This, it is felt, might be a representative situation, with the possible exception of letting $\sigma_{w_1} = 0$. This condition was brought about by the immediate lack of enough random function generators as well as a consideration of not investigating the effect of too many conditions at one time.

In Fig. 3 time histories are shown for the condition where $a_1(t)$, $a_2(t)$, and $a_3(t)$ are constants with respective values of -1, -1, and $+1$. The initial estimates of $\hat{a}_1(t)$, $\hat{a}_2(t)$, and $\hat{a}_3(t)$ are zero. The input $u^*(t)$ to the process is generated by a random noise generator with a bandwidth of approximately 2 cps. The noisy data $z(t)$ are recorded in channel 1, from which estimates of $a_1(t)$, $a_2(t)$, $a_3(t)$, $\hat{x}(t)$, and $dx(t)/dt$ are made. The estimates $\hat{x}(t)$, $\hat{a}_1(t)$, $\hat{a}_2(t)$, and $\hat{a}_3(t)$ are recorded in channels 3, 4, and 5, respectively, and the error in the estimate of $\hat{x}(t)$ is recorded in channel 6. It is observed that a good estimate of all parameters is made in less than one period of the natural frequency of the system. A similar situation is shown in Fig. 4 for a step input. Figure 5 illustrates an extreme condition when the initial estimates are completely reversed from their actual values. In Fig. 6 is shown the condition when $a_1(t)$ varies linearly with time. The broken line is the actual value of $a_1(t)$. Initial estimates of $a_2(t)$ and $a_3(t)$ are correct, whereas the initial estimate of $a_1(t)$ is zero. It should be observed that, although the process goes from a stable condition to an unstable condition, parameter identification continues without difficulty. A similar situation is shown in Fig. 7, where both $a_1(t)$ and $a_3(t)$ vary linearly with time, illustrating that the method is capable of recognizing the phenomenon of control reversal.

The remaining runs were made with simulation of the entire control system. The desired response was taken, unless otherwise specified, as that of a system with 0.7 critical damping. For this condition, the parameters $a_1^*(t)$, $a_2^*(t)$, and $a_3^*(t)$ which generate the desired response are -1.4, -1, and $+1$, respectively. A step input to the control system was used in most cases because one can quickly evaluate the systems response. Figure 8 illustrates the condition when $a_1(t)$, $a_2(t)$, and $a_3(t)$ are constants with the desired values of -1.4, -1, and $+1$, with initial estimates of the parameters all zero. The desired response $x^*(t)$ is recorded in channel 1, and the output $x(t)$ with additive noise which collectively represents the data $z(t)$ is recorded in channel 2. The estimates $\hat{a}_1(t)$, $\hat{a}_2(t)$, and $\hat{a}_3(t)$ are recorded in channels 3, 4, and 5. The estimation of $\hat{a}_1(t)$ takes considerably longer than that of $\hat{a}_2(t)$ and $\hat{a}_3(t)$, because, once the steady-state response to the step input (square wave input) is reached, no further information about the relative damping can be obtained. In Fig. 9 the desired response has been changed to 0.35 critical damping [$a_1^*(t) = 0.7$, $a_2^*(t) = -1$, $a_3^*(t) = +1$] with $a_1(t)$, $a_2(t)$, and $a_3(t)$ equal to -1.4, -1, and $+1$. Initial estimates $a_1(t)$, $a_2(t)$, and $a_3(t)$ are all zero. Figure 10 shows the condition when $a_3(t)$ varies linearly with time, with all initial estimates zero and a random input. It should be noted that the system can recognize the cope with control reversal.

The particular runs shown here represent only a few of the many runs made and were chosen to be representative of different conditions under which such a system might operate. A first-order system, with variable time constant and gain that was actually studied prior to the second-order system illustrated throughout this paper, was also simulated, with equally good results.

Conclusions

The primary purpose of this paper was to develop a statistical means of identifying the parameters of a linear system from noisy data and to show how the method might be applied to an adaptive control system. Although many approximations and assumptions were made, excellent results have been demonstrated by experiment.

If additional measurement data are available (x_2 in example discussed), **M** and Ψ become rectangular matrices, v becomes a column vector, and the variance σ_v^2 becomes a covariance matrix. The vector matrix equations remain the same.

The reader might feel that this particular demonstration is indeed trivial compared to a more general formulation that has been postulated. Certainly this is a valid concern, to which the following comment is offered. Observe that Eq. (1) is a nonlinear equation in the variables x_1, x_2, a_1, a_2, and a_3. One seeks to estimate these variables given data **z**, albeit postulating certain not unreasonable behavior of the elements a_1, a_2, and a_3. From this point of view, the reader is referred to the works of Battin[6] and others[7] which successfully use a similar technique of sequential linear regression to estimate the state of a six-dimensional nonlinear system (space vehicle trajectory determination). Actually, it was that application of Kalman's techniques which led the authors to consider the identification problem from the point of view expressed in this paper. The excellent results obtained by Battin et al. would seem to justify the authors' expectation of successful application of the proposed identification scheme to higher-order systems.

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Rise and Set Time of a Satellite about an Oblate Planet

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A technique is presented for the determination of the rise and set times of a satellite about an oblate planet. The solution depends on the ability to determine the rise and set eccentric anomalies from a single transcendental equation. The solution of the transcendental equation is facilitated greatly by an approximating technique. The cases of both zero and minimum elevation angles are analyzed. Numerical results with a strictly integrated orbit are presented for comparison with the closed-form solution.

THE purpose of this paper is to present a Keplerian closed-form solution to the rise and set time problem. In effect, this problem usually involves the calculation of the rise and set universal time of a given satellite from a specific ground station.

In the past, it has been the custom to solve the problem by letting the satellite run through its ephemeris and checking at each instant to see whether the elevation angle h of the satellite with respect to a ground station was greater than some minimum value. However, by attacking the problem from a different point of view, i.e., with the eccentric anomaly taken to be the independent variable, it is possible to obtain a closed-form solution to the satellite visibility problem. Specifically, the closed-form solution is a single transcendental equation in the eccentric anomalies corresponding to a rise and set time for a given orbital pass of a satellite. It is more difficult to solve the controlling equation than the standard Keplerian equation. However, the method offers the advantage that the controlling equation is solved only once per orbital period, as contrasted with the hundreds of times the Keplerian equation must be solved by the standard step-by-step technique.

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